

Distinct monomial orders with same induced orderings

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Abstract

We prove that the lexicographic, degree lexicographic and the degree reverse lexicographic orders for monomials in $R_n = K[X_1, \dots, X_n]$ are uniquely determined by their induced orderings, (i.e. their restrictions to $R_{n,i} = K[X_1, \dots, \hat{X}_i, \dots, X_n]$), when $n \geq 4$. We also show that for any $n \geq 4$ there are monomial orders that are not uniquely determined by their induced orderings, and provide examples of these orders for each n .

1 Introduction

Monomial orderings play a central role in computational commutative algebra, computational algebraic geometry and combinatorial commutative algebra because of the theory of Gröebner basis and their applications. The properties of the classic monomial orderings (lexicographic, graded lexicographic and reverse lexicographic) have allowed for many insights on Hilbert functions, Betti numbers and regularity of monomial ideals in statements regarding Lex-segments and Generic Initial Ideals. The books [2], [3] and [8] provide great introductions to these topics. Other results and advanced applications can be found in [5] and [7].

The graded lexicographic order is uniquely determined by its induced orderings when $n > 3$, a fact that is equivalent to a characterization of compressed ideals due to Jeff Mermin, [6, Theorem 3.12]. The property of being uniquely determined by induced orderings when $n > 3$ is also shared by the lexicographic and reverse lexicographic orders, we present proofs in Theorem 5. These facts prompted A. Conca [1] to ask whether there is $k \in \mathbb{N}$, such that any monomial order on R_n , $n > k$, is uniquely determined by its induced orderings. We answer Conca's question, Question 6, negatively in Theorem 12, and present examples of different monomial orderings with same induced orderings explicitly.

2 Monomial orders and their matrix representations

Let K be a field, $S_n = \{X_1, \dots, X_n\}$ and $R_n = K[S_n]$ be the polynomial ring in n variables, a monomial in R_n is of the form $X^\alpha = X_1^{\alpha_1} X_2^{\alpha_2} \dots X_n^{\alpha_n}$. We will abuse notation and depending on the context denote the monomial X^α by either $\underline{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)$ or $\underline{\alpha} = [\alpha_1 \ \alpha_2 \ \dots \ \alpha_n]^T$. The set of all monomials belonging to a set S will be denoted by $\text{Mon}(S)$.

Additionally we will denote the set of $m \times n$ matrices with entries in the set S by $M_{m \times n}(S)$ and the set of nonnegative integers, $\{0, 1, 2, \dots\}$, by \mathbb{N}^* .

Definition 1. A **monomial order** in R_n is a total order $<_\tau$ in the elements of $\text{Mon}(R_n)$ with the following two additional properties:

1. If $u \neq 1$ then $1 <_\tau u$ for all $u \in \text{Mon}(R_n)$ (or equivalently $(0, \dots, 0) <_\tau \underline{\alpha}$ for all $\underline{\alpha} \in (\mathbb{N}^*)^n$ with $\underline{\alpha} \neq (0, \dots, 0)$).
2. If $u, v \in \text{Mon}(R_n)$ are such that $u <_\tau v$ then $u + w <_\tau v + w$ for all $w \in \text{Mon}(R_n)$ (or equivalently if $\underline{\alpha} <_\tau \underline{\beta}$ then $\underline{\alpha} + \underline{\gamma} <_\tau \underline{\beta} + \underline{\gamma}$).

We will now give an overview of the three classical examples of monomial orders, for an in-depth study refer to [2, Chapter 2], [3, Chapter 1] or [8, Chapter 1].

Example 2. The **lexicographic order**, denoted $<_{lex}$. We say that $\underline{\alpha} <_{lex} \underline{\beta}$ if the leftmost nonzero entry of $\underline{\beta} - \underline{\alpha}$ is positive.

For example: $(2, 2, 2) <_{lex} (2, 3, 0) <_{lex} (3, 0, 3)$.

The **graded lexicographic order**, denoted $<_{deglex}$. We say that $\underline{\alpha} <_{deglex} \underline{\beta}$ if $\sum_{i=1}^n \alpha_i < \sum_{i=1}^n \beta_i$, or, if $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i$ and $\underline{\alpha} <_{lex} \underline{\beta}$.

For example: $(2, 3, 0) <_{deglex} (2, 2, 2) <_{deglex} (3, 0, 3)$.

The **reverse lexicographic order**, denoted $<_{revlex}$. We say that $\underline{\alpha} <_{deglex} \underline{\beta}$ if $\sum_{i=1}^n \alpha_i < \sum_{i=1}^n \beta_i$, or, if $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i$ and the rightmost nonzero entry of $\underline{\beta} - \underline{\alpha}$ is negative.

For example: $(2, 3, 0) <_{revlex} (3, 0, 3) <_{revlex} (2, 2, 2)$.

The graded lexicography and reverse lexicographic orders are examples of **graded monomial orders** (i.e. $\underline{\alpha} <_\tau \underline{\beta}$ whenever $\sum_{i=1}^n \alpha_i < \sum_{i=1}^n \beta_i$).

Remark 3. Due to the definition of monomial order it is easy to see that R_1 admits only one monomial order, namely $1 < x < x^2 < \dots$. So in this case the lexicographic, graded lexicographic and reverse lexicographic orders coincide in R_1 .

In R_2 there are infinitely many monomial orders, but R_2 admits only two graded monomial orders, as a consequence of Lemma 10. In this case the graded lexicographic

and reverse lexicographic orders actually coincide in R_2 , while the lexicographic order does not coincide with them anymore.

In R_3 all three classical orders are different, as it can be seen by the different ways in which the elements of the set $\{(2, 3, 0), (2, 2, 2), (3, 0, 3)\}$ were ordered before.

These observations seem to indicate that as n increases the classical monomial orders are growing further apart. To formalize what we mean by growing further apart we will use the concept of induced ordering.

Definition 4. Let $S_{n,i} = S_n - \{x_i\}$ and $R_{n,i} = K[S_{n,i}]$. The i^{th} induced ordering of $<_\tau$, denoted $<_{\tau,i}$, is the monomial order in $R_{n,i}$ with the property:

$$\underline{\alpha} = (\alpha_1, \dots, \alpha_{i-1}, 0, \alpha_{i+1}, \dots, \alpha_n) <_\tau (\beta_1, \dots, \beta_{i-1}, 0, \beta_{i+1}, \dots, \beta_n) = \underline{\beta} \text{ if and only if}$$

$$(\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_n) <_{\tau,i} (\beta_1, \dots, \beta_{i-1}, \beta_{i+1}, \dots, \beta_n).$$

From this point on instead of $(\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_n) <_{\tau,i} (\beta_1, \dots, \beta_{i-1}, \beta_{i+1}, \dots, \beta_n)$ we will write $(\alpha_1, \dots, \alpha_{i-1}, 0, \alpha_{i+1}, \dots, \alpha_n) <_{\tau,i} (\beta_1, \dots, \beta_{i-1}, 0, \beta_{i+1}, \dots, \beta_n)$.

The definition above allows us to state the following theorem.

Theorem 5. If $n > 3$ and $<_{\tau,i}$ is the lexicographic (resp. graded lexicographic, reverse lexicographic) order in $R_{n,i}$ for all $1 \leq i \leq n$ then $<_\tau$ is the lexicographic (resp. graded lexicographic, reverse lexicographic) in R_n .

Proof. We will first prove the case for the lexicographic order. Assume that $\underline{\alpha} <_{lex} \underline{\beta}$, and let $k = \min\{j : \alpha_j \neq \beta_j\}$.

- If $k > 1$ then $(\alpha_1, \dots, \alpha_{k-1}, \alpha_k, \dots, \alpha_n) <_\tau (\beta_1, \dots, \beta_{k-1}, \beta_k, \dots, \beta_n)$ if and only if $(\alpha_1, \dots, \alpha_{k-1}, \alpha_k, \dots, \alpha_n) <_{lex} (\beta_1, \dots, \beta_{k-1}, \beta_k, \dots, \beta_n)$ since $(0, \dots, \alpha_k, \dots, \alpha_n) <_{lex,1} (0, \dots, \beta_k, \dots, \beta_n)$ and $<_{\tau,1} = <_{lex,1}$.
- If $k = 1$ then $(\alpha_1, \dots, \alpha_n) <_\tau (\beta_1, \dots, \beta_n)$ if and only if $(\alpha_1, \dots, \alpha_n) <_{lex} (\beta_1, \dots, \beta_n)$, due to the sequence $(\alpha_1, \alpha_2, \dots, \alpha_n) \leq_{lex,1} (\alpha_1, \sum_{i=2}^n \alpha_i, \dots, 0) <_{lex,n} (\beta_1, \dots, 0) \leq_{lex,1} (\beta_1, \dots, \beta_n)$ and the fact that $<_{\tau,i} = <_{lex,i}$ for all $1 \leq i \leq n$.

For the case of the reverse lexicographic order notice that if there is i such that $\alpha_i = \beta_i$ then $\underline{\alpha} <_\tau \underline{\beta}$ if and only if $(\alpha_1, \dots, \alpha_{i-1}, 0, \alpha_{i+1}, \dots, \alpha_n) <_\tau (\beta_1, \dots, \beta_{i-1}, 0, \beta_{i+1}, \dots, \beta_n)$, which is equivalent to saying $(\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_n) <_{\tau,i} (\beta_1, \dots, \beta_{i-1}, \beta_{i+1}, \dots, \beta_n)$, and this is equivalent to $(\alpha_1, \dots, \alpha_{i-1}, 0, \alpha_{i+1}, \dots, \alpha_n) <_{revlex} (\beta_1, \dots, \beta_{i-1}, 0, \beta_{i+1}, \dots, \beta_n)$, since $<_{\tau,i} = <_{revlex,i}$, and this happens if and only if $\underline{\alpha} <_{revlex} \underline{\beta}$.

Hence $(d, 0, \dots, 0) <_\tau (0, 0, \dots, d+1)$ if and only if $(d, 0, \dots, 0) <_{revlex} (0, 0, \dots, d+1)$.

Additionally if $\gamma_i = \min(\alpha_i, \beta_i)$ then $\underline{\alpha} < \underline{\beta}$ if and only if $\underline{\alpha} - \underline{\gamma} < \underline{\beta} - \underline{\gamma}$ for any monomial order $<$.

Because of the observations above, to guarantee that $<_\tau = <_{revlex}$ it is enough to prove that $\underline{\alpha} <_\tau \underline{\beta}$ if and only if $\underline{\alpha} <_{revlex} \underline{\beta}$ for all $\underline{\alpha}, \underline{\beta}$ such that $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i = d$, $\alpha_i \cdot \beta_i = 0$ and $\alpha_i + \beta_i > 0$. So we will restrict ourselves to these cases from this point on.

Let's assume without loss of generality that $\alpha_n \neq 0$, and let $k = \max\{j : \beta_j \neq 0\}$, so $k < n$. We will divide the argument in five cases:

- *Case 1:* $k < n - 1$

Notice that $(\alpha_1, \dots, 0, \alpha_{k+1}, \dots, \alpha_{n-1}, \alpha_n) <_{\text{revlex}} (\beta_1, \dots, \beta_k, 0, \dots, 0, 0)$ if and only if $(\alpha_1, \dots, 0, \alpha_{k+1}, \dots, \alpha_{n-1}, \alpha_n) <_{\tau} (\beta_1, \dots, \beta_k, 0, \dots, 0, 0)$, due to the sequence:

$$(\alpha_1, \dots, 0, \alpha_{k+1}, \dots, \alpha_{n-1}, \alpha_n) <_{\text{revlex}, k} (0, \dots, 0, 0, \dots, d, 0)$$

$$(0, \dots, 0, 0, \dots, d, 0) <_{\text{revlex}, n} (0, \dots, d, 0, \dots, 0, 0) \leq_{\text{revlex}, n} (\beta_1, \dots, \beta_k, 0, \dots, 0, 0)$$

and the fact that $<_{\tau, i} = <_{\text{revlex}, i}$ for all $1 \leq i \leq n$.

- *Case 2:* $k = n - 1$ and $\alpha_i \neq 0$ for all $i < n - 1$

Notice that $(\alpha_1, \dots, \alpha_{n-2}, 0, \alpha_n) <_{\text{revlex}} (0, \dots, 0, d, 0)$ if and only if $(\alpha_1, \dots, \alpha_{n-2}, 0, \alpha_n) <_{\tau} (0, \dots, 0, d, 0)$, due to the sequence:

$$(\alpha_1, \dots, \alpha_{n-2}, 0, \alpha_n) <_{\text{revlex}, n-1} (\alpha_1 + \alpha_{n-2}, \dots, 0, 0, \alpha_n) <_{\text{revlex}, n-2} (0, \dots, 0, d, 0)$$

and the fact that $<_{\tau, i} = <_{\text{revlex}, i}$ for all $1 \leq i \leq n$.

- *Case 3:* $k = n - 1$, $\alpha_i = 0$ for all $i < n - 1$.

Notice that $(0, \dots, 0, 0, d) <_{\text{revlex}} (\beta_1, \dots, \beta_{n-2}, \beta_{n-1}, 0)$ if and only if $(0, \dots, 0, 0, d) <_{\tau} (\beta_1, \dots, \beta_{n-2}, \beta_{n-1}, 0)$, due to the sequence:

$$(0, \dots, 0, 0, d) <_{\text{revlex}, 1} (0, \dots, 0, d, 0) <_{\text{revlex}, n} (\beta_1, \dots, \beta_{n-2}, \beta_{n-1}, 0)$$

and the fact that $<_{\tau, i} = <_{\text{revlex}, i}$ for all $1 \leq i \leq n$.

- *Case 4:* $k = n - 1$, $\alpha_{n-2} = 0$ and there is $l < n - 2$ with $\alpha_l \neq 0$.

Notice that $(\alpha_1, \dots, \alpha_l, 0, \dots, 0, 0, \alpha_n) <_{\text{revlex}} (\beta_1, \dots, 0, \beta_{l+1}, \dots, \beta_{n-2}, \beta_{n-1}, 0)$ if and only if $(\alpha_1, \dots, \alpha_l, 0, \dots, 0, 0, \alpha_n) <_{\tau} (\beta_1, \dots, 0, \beta_{l+1}, \dots, \beta_{n-2}, \beta_{n-1}, 0)$, due to the sequence:

$$(\alpha_1, \dots, \alpha_l, 0, \dots, 0, 0, \alpha_n) <_{\text{revlex}, n-2} (0, \dots, 0, d, 0) <_{\text{revlex}, n} (\beta_1, \dots, 0, \beta_{l+1}, \dots, \beta_{n-2}, \beta_{n-1}, 0)$$

and the fact that $<_{\tau, i} = <_{\text{revlex}, i}$ for all $1 \leq i \leq n$.

- *Case 5:* $k = n - 1$, $\alpha_{n-2} \neq 0$ and there is $l < n - 2$ with $\alpha_l = 0$.

Notice that $(\alpha_1, \dots, 0, \dots, \alpha_{n-2}, 0, \alpha_n) <_{\text{revlex}} (\beta_1, \dots, \beta_l, \dots, 0, \beta_{n-1}, 0)$ if and only if $(\alpha_1, \dots, 0, \dots, \alpha_{n-2}, 0, \alpha_n) <_{\tau} (\beta_1, \dots, \beta_l, \dots, 0, \beta_{n-1}, 0)$, due to the sequence:

$$(\alpha_1, \dots, 0, \dots, \alpha_{n-2}, 0, \alpha_n) <_{\text{revlex}, l} (0, \dots, 0, \dots, 0, d, 0) <_{\text{revlex}, n} (\beta_1, \dots, \beta_l, \dots, 0, \beta_{n-1}, 0)$$

and the fact that $<_{\tau, i} = <_{\text{revlex}, i}$ for all $1 \leq i \leq n$.

The case for the graded lexicographic order is analogous to that of the reverse lexicographic order.

This completes the proof. □

Theorem 5, coupled with Remark 3, motivated A. Conca, [1], to ask the following:

Question 6. Is there $n > 3$ such that any monomial ordering in R_n is uniquely determined by its induced orderings?

The answer to this question is negative, and its proof is given in Theorem 12. Furthermore the answer is still negative if we just focus our attention on graded monomial orders.

Theorem 7. [Robbiano] *Given a monomial order $<_\tau$ in R_n there exists $A \in M_{m \times n}(\mathbb{R})$ such that $\underline{\alpha_1} <_\tau \underline{\alpha_2}$ if and only if $A \cdot \underline{\alpha_1} <_{lex} A \cdot \underline{\alpha_2}$.*

Proofs of this result can be found in [4], [9], and [10].

Example 8. The $n \times n$ identity matrix I corresponds to the $<_{lex}$ order in R_n .

$$\text{While the matrices } G = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \text{ and } R = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

correspond to $<_{deglex}$ and $<_{revlex}$ respectively.

We provide now a converse of Theorem 7, which is presented as a statement that encompasses [3, Exercise 2.8] and [8, Proposition 1.4.12]

Lemma 9. *Let $A \in M_{m \times n}(\mathbb{R})$ such that $\text{Ker}(A) \cap \mathbb{Z}^n = \{(0, \dots, 0)\}$ and the first non-zero entry in each column is positive. Then there is a monomial order $<_A$ in R_n with $\underline{\alpha_1} <_A \underline{\alpha_2}$ if $A \cdot \underline{\alpha_1} <_{lex} A \cdot \underline{\alpha_2}$.*

In particular if $A \in M_{n \times n}(\mathbb{N}^*)$ and $\det(A) \neq 0$ then $<_A$ is a monomial order.

Lemma 10. *Let $A, B \in M_{m \times n}(\mathbb{R})$ be matrices defining monomial orders and let $L \in M_{m \times m}(\mathbb{R})$ be a lower triangular matrix with positive entries in the diagonal such that $B = LA$. Then the monomial orders defined by A and B are the same.*

An elementary proof of Lemma 10 appears in [9], it is also an exercise [8, Tutorial 9].

Additionally if we have a matrix representation for a monomial order $<_\tau$ in terms of an $n \times n$ matrix with entries on the integers, we can find the matrix representation for its i^{th} induced ordering because of the following proposition in [8, Proposition 1.4.13].

Lemma 11. *If A is an $n \times n$ matrix representing the monomial order $<_\tau$ then its i^{th} induced ordering $<_{\tau, i}$ is represented by the matrix A_i which is obtained by first deleting the i^{th} column of A and then the first row which is linearly independent on those above it.*

We are now ready to prove our main result.

Theorem 12. For $n \geq 4$ there exist $<_{\tau}$ and $<_{\tau'}$ distinct monomial orders in R_n such that their induced orderings $<_{\tau,i}$ and $<_{\tau',i}$ are the same in $R_{n,i}$ for all $1 \leq i \leq n$.

Proof. Consider the $n \times n$ matrices, $n \geq 4$,

$$C_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ \frac{n^2+n+2}{2} & n-1 & n-2 & \cdots & 4 & 3 & 2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$D_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ \frac{n^2-n}{2} & n-1 & n-2 & \cdots & 4 & 2 & 3 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix}.$$

A simple reduction allows us to calculate $\det(C_n) = 4 - 3n$ and $\det(D_n) = 5 - 2n$. Since $4 - 3n \neq 0 \neq 5 - 2n$ for any value of $n > 3$, Lemma 9 proves that they both define monomial orders on R_n .

Notice that the monomial orders defined by C_n and D_n are distinct in R_n , since $(2, 0, \dots, 0, n, n^2, 2) <_{C_n} (4, 0, \dots, 0, n^2, n, 1), (4, 0, \dots, 0, n^2, n, 1) <_{D_n} (2, 0, \dots, 0, n, n^2, 2)$.

Let $C_{n,i}$ (resp. $D_{n,i}$) be the $(n-1) \times (n-1)$ matrix obtained by eliminating the i^{th} column and the n^{th} row of C_n (resp. D_n). We will now present the numerical values of the determinants of $C_{n,i}$ and $D_{n,i}$:

$$\begin{aligned} \det(C_{n,1}) &= (-1)^n & \det(D_{n,1}) &= (-1)^n \cdot 2 \\ \det(C_{n,i}) &= (-1)^{n+i-2} & \det(D_{n,i}) &= (-1)^{n+i-2} \cdot 2 \\ \det(C_{n,n-1}) &= -2n & \det(D_{n,n-1}) &= -1 - 2n \\ \det(C_{n,n}) &= 4 - 3n & \det(D_{n,n}) &= 5 - 2n \end{aligned}$$

for $2 \leq i \leq n-2$.

All these determinants are nonzero, for $n > 3$, so Lemma 11 implies that $C_{n,i}$ (resp. $D_{n,i}$) are matrix representations for the i^{th} induced ordering of $<_{C_n}$ (resp. $<_{D_n}$) in $R_{n,i}$.

Finally the facts that $C_{n,i}$ and $D_{n,i}$ are invertible and their first $n-2$ rows are the same guarantee that there is a lower triangular matrix

$$U_i = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ a_{i,1} & a_{i,2} & \cdots & a_{i,n-2} & a_{i,n-1} \end{bmatrix},$$

such that $D_{n,i} = U_i C_{n,i}$. By Cramer's rule $a_{i,n-1} = \frac{\det(\bar{D}_{n,i})}{\det(C_{n,i})} > 0$ which implies by Lemma 10, that the induced orders $C_{n,i}$ and $D_{n,i}$ are the same for all $1 \leq i \leq n$.

This concludes our proof. \square

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